Matrix Representations and Similarity

1. Recall:

Let V and V' be vector spaces with ordered basis $B = (b_1, b_2, \ldots b_n)$ and $B' = (b_1', b_2', \ldots b_n')$, respectively. Then, the matrix rep of T relative to B, B' is denoted by $R_{B,B'}$ and is given by

$$R_{B,B'} = \begin{bmatrix} | & | & & | \\ T(b_1)_{B'} & T(b_2)_{B'} & \cdots & T(b_n)_{B'} \\ | & | & & | \end{bmatrix}$$

where $T(b_i)_{B'}$ is the coordinate vector of $T(b_i)$ relative to B'. Furthermore, $R_{B,B'}$ is the unique matrix satisfying

$$T(v)_{B'} = R_{B,B'} V_B$$

To find $R_{B,B'}$, we need

1. $M_{T(b)}$, the $m \times n$ matrix whose col vectors are $T(b)$.
2. $M_{B'}$

$$\left[M_{B'} \mid M_{T(b)} \right] \sim \left[I \mid R_{B,B'} \right]$$

2. Multiplicative Property of Matrix Reps:

Let $T: R^n \to R^m$ and $T': R^m \to R^s$ be 2 linear transformations. Then, Matrix for $(T' \circ T)$ = (Matrix for T')(Matrix for T).

3. Relationship Between Matrix Rep of Linear Transformations and Change of Bases Matrix:

Let B and B' be ordered bases for R^n.

$$R_{B'} = C_{B,B'} \cdot R_B \cdot C_{B',B}$$
$$= C^{-1} R_B C$$

Consequently, $R_{B'}$ and R_B are similar matrices.

Thms

2 nxn matrices are similar iff they are matrix reps of the same linear transformation T relative to suitable ordered basis.

Similar matrices have the same eigenvalues.

4. Eigenvalues and EigenVectors of Similar Matrices:

Let A and R be similar nxn matrices s.t. $R = C^{-1}AC$. Let the eigenvalues of A be $\lambda_1, \lambda_2, \ldots \lambda_n$. Note, the eigenvalues don't have to be distinct. Then,

1. The eigenvalues of R are also $\lambda_1, \lambda_2, \ldots \lambda_n$.
2. The algebraic and geometric multiplicity of each λ_i are the same for A and R.

3. If V_i in R^n is an eigenvector of A corresponding to λ_i, then $C^{-1}V_i$ is an eigenvector of R corresponding to λ_i.

Recall:

The algebraic multiplicity of an eigenvalue is how many times that eigenvalue appears.

E.g.

Suppose $0 = (\lambda-3)^2(\lambda+4)$. Then, $\lambda_1 = 3$ and $\lambda_2 = -4$. However, since 3 occurs twice, the algebraic multiplicity of $\lambda_1 = 2$, while the algebraic multiplicity of $\lambda_2 = 1$.

The geometric multiplicity of an eigenvalue is the nullspace of its eigenspace.

E.g.

Suppose $E_{\lambda_1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Then, the nullspace of $E_{\lambda_1} = 2$, so its geometric multiplicity is 2.

If an eigenvalue's geo = its alge, then A is diagonizable.

Note: An eigenvalue's geo is always $\leq$ its alge.

5. Diagonalization:

A linear transformation of a finite-dimensional vector space V is diagonalizable if V has an ordered basis consisting of eigenvectors of T.

E.g.

Consider the vector space P_2 of all polynominals of degree at most 2 and let B' be the ordered basis $(1, x, x^2)$ for P_2. Let $T: P_2 \to P_2$ be the lin trans s.t.

$T(1) = 3 + 2x + x^2$

$T(x) = 2$

$T(x^2) = 2x^2$

Find $T^4(x+2)$

Solution:

$$R_{B'} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

To find the eigenvalues of $R_{B'}$, I do

$$0 = \det(R_{B'} - \lambda I)$$

$$= \left| \begin{bmatrix} 3-\lambda & 2 & 0 \\ 2 & -\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \right|$$

$$= (3-\lambda)\left[(-\lambda)(2-\lambda)\right] - 2\left[(2)(2-\lambda)\right]$$
$$= (3-\lambda)\left[-2\lambda+\lambda^2\right] - 2\left[4-2\lambda\right]$$
$$= -6\lambda+3\lambda^2+2\lambda^2-\lambda^3-8+4\lambda$$
$$= -\lambda^3+5\lambda^2-2\lambda-8$$

$\lambda_1=-1$, $\lambda_2=2$, $\lambda_3=4$

When $\lambda_1=-1$

$R_{B'} - \lambda_1 I$

$$= \begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & -6 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $x_3=s$

$x_1+3x_3=0$

$x_1=-3x_3$

$=-3s$

$x_2-6x_3=0$

$x_2=6x_3$

$=6s$

The eigenvector of λ_1 is $[-3, 6, 1]$.

$P(\lambda_1) = -3+6x+x^2$

When $\lambda_2 = 2$

$$R_{B'} - \lambda_2 I = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $x_3 = s$

The eigenvector of $\lambda_2 = [0,0,1]$.

$P(\lambda_2) = x^2$

When $\lambda_3 = 4$

$$R_{B'} - \lambda_3 I = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $x_3 = s$
Then, the eigenvector of λ_3 is $[2, 1, 1]$.
$P(\lambda_3) = 2 + x + x^2$

Let B be the ordered basis $(-3+6x+x^2, x^2, 2+x+x^2)$.
The coordinate vector d of $x+2$ relative to the basis B is $[0, -1, 1]$.

$$\left[\begin{array}{ccc|c} -1 & 0 & 2 & 2 \\ 6 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$d_0 = 0$
$d_1 = -1$
$d_2 = 1$

$$\sim \left[\begin{array}{ccc|c} -1 & 0 & 2 & 2 \\ 6 & 0 & 1 & 1 \\ 0 & 1 & 3 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 13 & 13 \\ 6 & 0 & 1 & 1 \\ 0 & 1 & 3 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 6 & 0 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 6 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Then, $T^k(x+2) = (\lambda_1)^k (d_0)(-3+6x+x^2) + (\lambda_2)^k (d_1)(x^2)$
$+ (\lambda_3)^k (d_2)(2+x+x^2)$

$$T^4(x+2) = 2^4(-1)(x^2) + 4^k(1)(2+x+x^2)$$
$$= -16x^2 + 256(2+x+x^2)$$
$$= 240x^2 + 256x + 512$$